Minimum Impulse Guidance

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A linearized theory is developed for minimum fuel guidance in the neighborhood of a minimum fuel space trajectory. The thrust magnitude is unrestricted so that the thrust is applied impulsively on both the nominal trajectory and the neighboring optimal trajectories. The analysis allows for additional small midcourse impulses as well as for small changes in the magnitude, direction, and timing of the nominal impulses. The fuel is minimized by determining the trajectory which requires the minimum total velocity change when summed over all the impulses. The analysis is deterministic and applies to arbitrary time-varying gravitational fields. Three separate time-open problems are treated; rendezvous, orbit transfer, and orbit transfer with tangential nominal impulses.

Nomenclature

$\delta\bar{R}$ = position deviation
t_f = nominal final time
δt = time deviation
u_c = component of midcourse impulse in the critical plane
u_{nc} = component of midcourse impulse in the noncritical direction
$\bar{V}_T$ = velocity of target trajectory
$\bar{V}_I$ = velocity of nominal interception trajectory
$\overline{\Delta V}$ = nominal terminal impulsive velocity change
$\Sigma\delta V$ = total change in impulsive velocity cost

Introduction

THIS is the second of a series of papers on minimum fuel guidance of high-thrust rockets. The first paper[1] illustrated the general approach by treating the particular problem of guidance from a hyperbolic to a circular orbit. The succeeding papers are intended to generalize this approach to more general classes of guidance problems. This generalization will be carried out in several stages. The present paper will consider the general case of time-open impulsive guidance. Later papers will extend the analysis to finite thrust.

There is a well-developed theory for minimum fuel impulsive guidance.[2,3,4] However, these references consider only the case of an unpowered nominal trajectory. The nominal trajectory around which the analysis is linearized is a coasting arc. The present paper is intended to generalize these results to nominal trajectories containing one or more finite impulses. The analysis will consider three different problems. The first problem to be treated will be minimum fuel guidance for time-open rendezvous. The second problem will be time-open orbit transfer, and the third problem will be an important special case of the second, where one or more of the finite impulses is tangent to the velocity vector.

Mathematical Model

The analysis of the present paper is linearized about a nominal trajectory containing one, or more, finite impulsive velocity changes. This nominal trajectory must be an optimal trajectory minimizing the sum of the absolute magnitude of its impulses for transfer between its terminal states. The problem considered is the deterministic problem of determining the minimum impulse transfer from a given state in a close neighborhood of the nominal state at a given initial time to the terminal state with time open. The nominal trajectory may lie in a general time-varying gravitational field. The analysis is a first-order analysis neglecting second-order terms. It is analogous to the neighboring optimal guidance schemes developed for smooth optimization problems without corners. The problem is complicated by the possession of corners and the possibility of introducing additional impulses. However, the problem is simplified because it is a first-order analysis. In general, the problem will be to guide the vehicle from a given initial state at a given initial time to a final time in the near vicinity of the nominal terminal time. For the orbit transfer problem the final time may be allowed to become arbitrarily large; it may also be possible to extend the initial time arbitrarily far backwards in time.

Analysis

I. Time-Open Rendezvous

The key concept in analyzing minimum impulse guidance for time-open rendezvous is the concept of a noncritical direction. This concept was originally developed for use in interception problems rather than rendezvous[2,5] but is also useful in analyzing rendezvous. Consider the case where the nominal trajectory has a single finite impulse which accomplishes rendezvous at a nominal terminal time. If rendezvous were to be accomplished at a slightly earlier time δt, then the point at which rendezvous is accomplished must be displaced by the negative product of the target velocity vector and the time change;

$$\delta\bar{R} = -\bar{V}_T\delta t \qquad @t = t_f - \delta t \tag{1}$$

This position is reached by the intercepter at an earlier time than the nominal arrival time. If the trajectory of the intercepter were continued to the nominal arrival time, it would have the position given by Eq. (2) and shown on Fig. 1;

$$\delta\bar{R}_I = -\bar{V}_T\delta t + \bar{V}_I\delta t = -\overline{\Delta V}\delta t \qquad @t = t_f \tag{2}$$

This indicates that, if the intercepter will intercept a specified line in space at the nominal arrival time, then it will (to first order) also intercept the target at a somewhat earlier or later time. This specified line passes through the nominal arrival point and has the direction of the nominal finite impulse.

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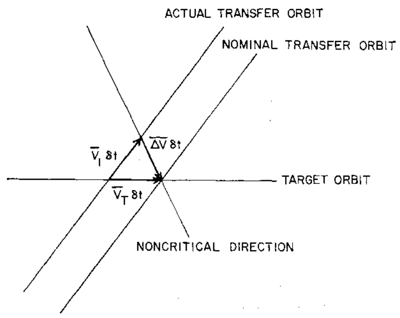

Fig. 1 Intercept geometry.

This direction through the nominal arrival point is known as the noncritical direction at the nominal arrival time. It represents the one permissible direction of position variation which will still lead to rendezvous. This noncritical direction may also be propagated backward in time by use of the state transition matrix. It will then define a noncritical direction at any point along the nominal trajectory.

In order to effect rendezvous, it is necessary to control the two components of position variation in the plane normal to the noncritical direction. This plane is known as the critical plane. Once the terminal position of the target vehicle and the rendezvous vehicle has been matched by reducing the position deviations in the critical plane to zero, rendezvous is accomplished by a finite impulse which nulls the difference between the target and intercepter velocities. To first order, only one component of terminal impulse variation adds linearly to the cost; that in the direction of the nominal impulse. Any small deviations in the velocity vector normal to this direction may be cancelled by small rotations of the nominal terminal impulse. Such rotations only increase cost to second order and may be neglected in a first-order analysis.

The foregoing considerations indicate that only two components of position and one component of velocity at the nominal final time must be controlled for time-open rendezvous. This reduces the original 6-dimensional parameter space to a 3-dimensional parameter space. If there is only one finite impulse, then the analysis for unpowered nominals in the references[2,3,4] may be applied without change to this 3-dimensional parameter space. That analysis indicates that the optimum solution has no more than three impulses. One of these impulses will represent a variation in the magnitude of the nominal impulse so that there are, at most, two midcourse impulses.

The required position correction at the nominal terminal time may be accomplished with a single midcourse impulse. If this corrective impulse occurs at a specified time, then the optimum direction of this impulse may easily be calculated. One component of the impulse will produce the position correction. This component will lie in the critical plane. There will also be a component of the midcourse impulse in the noncritical direction. This component will be used to reduce the magnitude of the large terminal impulse and will result in an over-all saving in impulse magnitude and fuel. The total change in impulsive velocity is given by Eq. (3);

$$\Sigma\delta V = (u_c^2 + u_{nc}^2)^{1/2} - (\partial|\overline{\Delta V}|/\partial u_{nc})u_{nc} - (\partial|\overline{\Delta V}|/\partial u_c)u_c \quad (3)$$

The optimum magnitude of the velocity component in the noncritical direction may be found by differentiating Eq. (3), and solving for the stationary minimum point given by Eq. (4);

$$u_{nc}^* = [(\partial|\overline{\Delta V}|/\partial u_{nc})|u_c|]/\{1 - [\partial|\overline{\Delta V}|/\partial u_{nc}]^2\}^{1/2} \quad (4)$$

The total cost of the optimum correction at a specified time is given by Eq. (5);

$$\Sigma\delta V^* = [1 - (\partial|\overline{\Delta V}|/\partial u_{nc})^2]^{1/2}|u_c| - (\partial|\overline{\Delta V}|/\partial u_c)u_c \quad (5)$$

In the particular case treated in Ref. 1 the midcourse correction should be made as early as possible and there will be only one midcourse impulse for the minimum fuel solution. This behavior will be typical of most cases as the time approaches the terminal time. However, in other cases as many as two midcourse impulses will be required to minimize the fuel consumption. It is also possible that a single impulse at a time later than the time under consideration may be optimum. There are both direct and indirect approaches to this optimization problem. The indirect method calculates the primer vector[6,7] from the direction given by the optimum direction of a single midcourse impulse at the current time to the terminal impulse at the terminal time. If this vector is less than unity at all intermediate points, then the single correction will be the absolute minimum fuel solution.

The direct method is a constructive approach utilizing the convex hull of the reachable set of terminal states.[2] This reachable set is constructed in a parameter space defined by the change in the terminal impulse magnitude and by the two position components in the terminal critical plane. Each of these parameters is normalized by the magnitude of the midcourse velocity change. An optimum maneuver must lie on the convex hull of the reachable sets in this space. The set of all impulse directions at a given time will define an ellipsoid in the parameter space. Equations (4) and (5) will define a generator of a cone which is tangent to the ellipsoid and whose apex is at minus one on the velocity axis (see Fig. 2). If a single correction at the earliest possible time is optimal, then the cones for all subsequent times will lie inside the initial cone. If two midcourse corrections are required, then the convex hull of all the cones will have a plane as one of its bounding surfaces. If a single correction at a later time is optimal, then one of the later cones will project through the cone corresponding to the initial time. The geometric construction for these cases may be reduced to a 2-dimensional construction by using the traces of the cone on the plane of the position variations. In exceptional cases where such traces do not produce closed figures, it may be necessary to use another plane that passes through the cones.

If the nominal trajectory contains one or more large impulses before the final impulse, then all necessary corrections may be made by utilizing small variations in these impulses. It is only necessary to consider small variations of timing and direction of these impulses. Such variations allow control of one component of position and two components of velocity at the time of the impulse. These three components may then be propagated to the terminal state by means of the state transition matrix. Except in exceptional cases it will be possible to control all three required components of the terminal state by this means. This control will (to first order) produce no increase in cost. This is shown by the fact that the primer vector passing through the two impulses of the optimal nominal trajectory is stationary with respect to small variations in impulse timing and direction.

II. Time-Open Orbit Transfer

If the object of the mission is orbit transfer rather than rendezvous, the particular phasing of the vehicle in the final

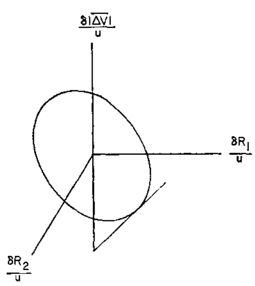

Fig. 2 Parameter space.

orbit is unspecified. This means that there will be a set of noncritical directions arising from all points on the target orbit in the vicinity of the nominal terminal time. This set of directions will to first order define a plane in which will lie the velocity vectors of both the target orbit and the transfer orbit at the nominal terminal time. All trajectories which are close neighbors of the nominal trajectory and which touch this noncritical plane at the nominal terminal time will also intersect the target trajectory at a time close to the nominal terminal time. For the orbit transfer problem it is only necessary to control one component of terminal position in the critical direction which is normal to the noncritical plane. The parameter space which must be considered is only 2-dimensional, containing one position component and one velocity component. There will be at most one midcourse impulse in addition to small variations in the terminal impulse. The optimum midcourse impulse may occur at a time other than the earliest possible time. In fact, in some cases this single midcourse impulse should occur in the neighborhood of the terminal orbit rather than in the neighborhood of the transfer orbit and at a time later than the time of the nominal terminal impulse. The latter case is easily analyzed by considering the set of reachable states in the vicinity of the terminal orbit, as well as in the vicinity of the transfer orbit.

III. Time-Open Orbit Transfer with Tangential Impulses

In many orbit transfer problems, such as the well-known Hohmann transfer, the impulses are applied tangent to the velocity vector. In such a case the noncritical plane of the preceding section becomes undefined and it is once again necessary to consider a 3-dimensional parameter space possessing two components of position variation. This case is similar to the case of time-open rendezvous and possesses a noncritical direction and a critical plane. As in the preceding section, it may be desirable to consider midcourse impulses in the terminal orbit as well as in the transfer orbit. It is possible to have a midcourse impulse before the major transfer impulse in the neighborhood of the transfer orbit, as well as a post-terminal time midcourse impulse in the neighborhood of the nominal terminal orbit. If there are one or more large impulses on the nominal trajectory before the terminal impulse, then variations in the timing and direction of these impulses may be used to control the trajectory. In the particular case of a Hohmann transfer, these variations will not be sufficient to control all out-of-plane deviations because the two impulses are located at singularities of the state transition matrix. In this case it will be necessary to utilize midcourse impulses in either the transfer orbit or one of the terminal orbits for controlling the out-of-plane component of the terminal position variation.

Conclusions

1) Minimum impulse time-open rendezvous in the neighborhood of an optimal nominal trajectory requires at most two small midcourse impulses if the nominal trajectory possesses one large finite impulse. Two midcourse impulses may be required if either the nominal trajectory or the deviations from it are nonplanar. If both the trajectory and deviations are planar, not more than one midcourse impulse will be required to realize minimum total impulse.

2) Minimum fuel, time-open orbit transfer in the near vicinity of an optimum nominal requires at most one small midcourse impulse if the nominal trajectory contains at least one finite impulse which is not tangent to the velocity vector. If both the nominal trajectory and the small deviations from it lie in the same plane, there will be no small midcourse impulse. In the latter case, the first-order minimum fuel solution will be a single impulse at the intersection of the two orbits.

3) For both time-open rendezvous and orbit transfer with two or more finite impulses, no midcourse impulse will be required unless the finite impulses occur at singularities of the state transition matrix.

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